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Granular reducts of formal fuzzy contexts

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ABSTRACT

Knowledge reduction is one of the key issues in knowledge discovery and data mining. During the construction of a concept lattice, it has been recognized that computational complexity is a major obstacle in deriving all the concept from a database. In order to improve the computational efficiency, it is necessary to preprocess the database and reduce its size as much as possible. Focusing on formal fuzzy contexts, we introduce in the paper the notions of granular consistent sets and granular reducts and propose granular reduct methods in the sense of reducing the attributes. With the proposed approaches, the attributes that are not essential to all the object concepts can be removed without loss of knowledge and, consequently, the computational complexity of constructing the concept lattice is reduced. Furthermore, the relationship between the granular reducts and the classification reducts in a formal fuzzy context is investigated.

1. Introduction

The theory of formal concept analysis (FCA), proposed by Wille [11,47], centers on the study of formal concepts and conceptual hierarchies. It has been employed to unravel, from relational information systems, hierarchical concepts organized as a lattice. Since its inception, FCA has been applied to many real-life problems including data mining, information retrieval, knowledge acquisition, software engineering, data base management systems and on other disciplines [8,10,18,22–25,28,33,45,57]. Over the years, FCA has been an important research area with appealing theoretical and practical issues.

FCA is formulated based on a formal context materialized as a set of objects, a set of attributes, and a binary relation usually taking the form of a binary table that relates the objects to the attributes with value 0 and 1. However, in many real-life problems, the binary relations are fuzzy rather than crisp. Thus, formal fuzzy contexts are more common than their crisp counterparts. For this reason, binary fuzzy relations are used to analyze Galois connections between objects and attributes. Burusco and Fuentes-Gonzáles [4] first examined FCA in a fuzzy setting, and they defined L-fuzzy concepts using implication operators. In recent years, many researchers have extended FCA theory by using the ideas from fuzzy logic reasoning or fuzzy set theory, and several generalizations of formal fuzzy concept have been made (please see [1,9,12,13,15,34,40,48]). On the other hand, Krajči [19] and Yahia et al. [53] independently proposed the "one-sided fuzzy concept", where each fuzzy concept (X, B) takes the form of "X is crisp and B is fuzzy", or "X is fuzzy and B is crisp". Zhang et al. [60] further constructed the "variable threshold concept lattices", i.e. crisp-fuzzy variable threshold concepts and fuzzy-crisp variable threshold concepts, in which the "one-sided fuzzy concept" becomes a particular case (threshold being equal to 1). One can refer [2] for a comprehensive survey and comparison of the existing approaches for fuzzy concept lattices.

Granular computing (GrC) is an approach for knowledge representation and data mining. A granule is a clump of objects (points) drawn together by some criteria. The main directions in the study of GrC are the construction of granules and computation with granules. The former deals with the formation, representation, and interpretation of granules, while the latter handles the utilization of granules in problem solving [38,51,54,55]. More recently, there has been an increasing interest in the study of GrC, and many methods and models have been proposed and studied [6,29,32,37,39,41,42,49,52,56,59].

It should be noted that a concept lattice is constructed by all the formal concepts combined with a hierarchical order of the concepts. At the bottom of a concept lattice structure are object concepts, and other concepts (contained in the concept lattice) can be represented as a join of some object concepts. Hence, the object concepts play an important role in the construction of concept

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lattice and can be viewed as a basic information granule in the concept lattice. Thus, concept lattice can be treated as a concrete model of GrC. Wu et al. [50] first examined the granular structure of concept lattices and applied it to knowledge reduction of formal contexts. Kang et al. [16,17] introduced GrC into FCA and ontology learning, and presented a unified model for concept-lattice building and rule extraction under fuzzy granularity and ontology model building, ontology merging and ontology connection at different levels of granulation.

Attribute reduction is an important issue in the discovery of knowledge in information systems. In terms of a formal context, attribute reduction searches for a minimal attribute subset that preserves the required properties. Interest in attribute reduction in FCA has rapidly increased in recent years [7,9,20,21,26,30,35,43,44,46,50,61]. Ganter and Wille [11] first introduced the notions of reducible attributes and reducible objects by reducing columns and rows in a formal context. Zhang and Wei [61] discussed attribute reduction in classical formal contexts, and formulated a reduction approach by using discernibility matrices and Boolean functions. Liu et al[31]. presented a reduction method for concept lattices based on rough set theory. In [46], Wang and Zhang proposed a reduction approach by keeping the meet-irreducible elements. Based on fuzzy K-means clustering, Kumar and Srinivas [20] put forward a method to reduce the size of a concept lattice by employing the corresponding object-attribute matrix. Shao et al. [44] formulated a knowledge-lossless approach to complexity reduction in formal decision contexts with which the complexity of concept lattice is substantially reduced. Li et al. [21] developed a rule-acquisition oriented framework of knowledge reduction for real decision formal contexts and formulated a reduction method by constructing a discernibility matrix and its associated Boolean function. Nevertheless, the aforementioned studies are carried out within classical formal contexts.

Based on the Lukasiewicz implication, Elloumi et al. [9] formulated a multi-level conceptual data reduction approach via the reduction of the object sets by keeping only the minimal rows in a formal fuzzy context. Belohlavek et al. [3] proposed a method to reduce the number of formal fuzzy concepts by keeping the so-called crisply generated fuzzy concepts derived from some crisp subset of attributes and leaving out non-crisply generated fuzzy concepts. Li and Zhang [27] introduced the notion of δ-reducts in formal fuzzy contexts, and gave some equivalent characterizations of the δ-consistent sets to determine δ-reducts. Comparing with the studies on knowledge reduction in the classical formal contexts, very little effort has been made to investigate the issue within formal fuzzy contexts. In a concept lattice, object concepts are actually more important, since every formal concept in a concept lattice can be represented as a join of some object concepts. Wu et al. [50] studied granular reducts in classical formal contexts by keeping the extensions of all object concepts. However, it should be noted that the number of formal concepts in a formal fuzzy context dramatically increases, making the structure of the corresponding lattice more complicated than those of a classical formal context. Thus, the reductions made in formal fuzzy contexts become more meaningful.

In this paper, we study granular reducts of formal fuzzy contexts, a generalization of those [50] in fuzzy framework. Accordingly, we propose some granular reduct approaches and investigated the relation between granular reducts and classification reducts in a formal fuzzy context. Specifically, we review in the next section some basic notions and properties of crisp-fuzzy concepts, and then analyze the basic structures of information granules and concept lattices derived from a formal fuzzy context and its sub-contexts. Furthermore, we present some theorems for judging join-irreducible elements in a concept lattice constructed from crisp-fuzzy concepts. In Section 3, we study the issue of granular reducts in formal fuzzy contexts. In Section 4, we propose some granular reduct approaches in consistent formal fuzzy decision contexts. The relationship between granular reduct and classification reduct in a formal fuzzy context is investigated in Section 5. The paper is then concluded with a summary and outlook for further research.

2. Preliminaries

In this section, we recall the notion of a fuzzy concept lattice constructed from crisp-fuzzy concepts and some of its main properties. More details can be found in [53] on crisp-fuzzy concepts.

2.1. Formal fuzzy contexts and crisp-fuzzy formal concepts

Yahia [53] and Krajči [19] independently proposed the “crisp-fuzzy concept”. In the following, we introduce its basic notion and investigate some of its properties used in our subsequent discussion.

Let U be a finite and nonempty set called the universe of discourse. We denote by $\mathcal{P}(U)$ and $\mathcal{F}(U)$ the set of all ordinary subsets of U and the set of all fuzzy sets in universe U, respectively.

For any $\widetilde{X}_1, \widetilde{X}_2 \in \mathcal{F}(U)$, $\widetilde{X}_1 \subseteq \widetilde{X}_2$ if and only if $\widetilde{X}_1(x) \le \widetilde{X}_2(x)$ $(\forall\ x \in U)$, and operations $\cup$ and $\cap$ on $\mathcal{F}(U)$ are defined by:

$$(\widetilde{X}_1 \cup \widetilde{X}_2)(x) = \widetilde{X}_1(x) \vee \widetilde{X}_2(x),$$
$$(\widetilde{X}_1 \cap \widetilde{X}_2)(x) = \widetilde{X}_1(x) \wedge \widetilde{X}_2(x).$$

The basic data set of FCA is a formal context. A formal fuzzy context is a triple $(U, A, \widetilde{I})$, where U and A are the object set and attribute set respectively, and $\widetilde{I} \in \mathcal{F}(U \times A)$ is a binary fuzzy relation between U and A.

Definition 1 [53]. Let $(U, A, \widetilde{I})$ be a formal fuzzy context. For $X \in \mathcal{P}(U)$, $\widetilde{B} \in \mathcal{F}(A)$, the operators $f : \mathcal{P}(U) \to \mathcal{F}(A)$ and $g : \mathcal{F}(A) \to \mathcal{P}(U)$ are defined respectively as follows:

$$f(X)(a) = \bigwedge_{x\in X} \widetilde{I}(x, a),\ \ a \in A,$$
$$g(\widetilde{B}) = \{x \in U | \forall\ a \in A, \widetilde{B}(a) \le \widetilde{I}(x, a)\}. \tag{1}$$

For any $x \in U$, for simplicity, we will write $f(x)$ instead of $f(\{x\})$.

Operators f and g form a Galois connection between $\mathcal{P}(U)$ and $\mathcal{F}(A)$, and the following properties can be obtained.

Property 1 [53]. *Let $(U, A, \widetilde{I})$ be a formal fuzzy context, $X, X_1, X_2, X_i \in \mathcal{P}(U)$, and $\widetilde{B}, \widetilde{B}_1, \widetilde{B}_2, \widetilde{B}_i \in \mathcal{F}(A)$, $i \in J$ (J is an index set). Then*

(1) $X_1 \subseteq X_2 \Rightarrow f(X_2) \subseteq f(X_1)$, $\widetilde{B}_1 \subseteq \widetilde{B}_2 \Rightarrow g(\widetilde{B}_2) \subseteq g(\widetilde{B}_1)$;
(2) $X \subseteq g \circ f(X)$, $\widetilde{B} \subseteq f \circ g(\widetilde{B})$;
(3) $f(X) = f \circ g \circ f(X)$, $g(\widetilde{B}) = g \circ f \circ g(\widetilde{B})$;
(4) $f(\bigcup_{i\in J} X_i) = \bigcap_{i\in J} f(X_i)$, $g(\bigcup_{i\in J} \widetilde{B}_i) = \bigcap_{i\in J} g(\widetilde{B}_i)$.

For a formal fuzzy context $(U, A, \widetilde{I})$, a pair $(X, \widetilde{B}) \in \mathcal{P}(U) \times \mathcal{F}(A)$ satisfying $X = g(\widetilde{B})$ and $\widetilde{B} = f(X)$ is called a crisp-fuzzy concept of $(U, A, \widetilde{I})$ (see [53]). For a set of objects $X \in \mathcal{P}(U)$ and a fuzzy set of attributes $\widetilde{B} \in \mathcal{F}(A)$, from Property 1 (3), we can observe that both $(g{\circ}f(X), f(X))$ and $(g(\widetilde{B}), f \circ g(\widetilde{B}))$ are crisp-fuzzy concepts. In particular, $(g{\circ}f(x), f(x))$ is a crisp-fuzzy concept for each $x \in U$ and is called an object concept. For two crisp-fuzzy concepts $(X_1, \widetilde{B}_1)$ and $(X_2, \widetilde{B}_2)$, we define $(X_1, \widetilde{B}_1) \le (X_2, \widetilde{B}_2)$ if and only if $X_1 \subseteq X_2$ (or equivalently, $\widetilde{B}_2 \subseteq \widetilde{B}_1$). All crisp-fuzzy concepts of $(U, A, \widetilde{I})$ form a complete lattice, denoted as $L(U, \widetilde{A}, \widetilde{I})$, in which the infimum and the supremum are defined respectively as follows:

$$(X_1, \widetilde{B}_1) \wedge (X_2, \widetilde{B}_2) = (X_1 \cap X_2, f \circ g(\widetilde{B}_1 \cup \widetilde{B}_2))$$
$$= (X_1 \cap X_2, f(X_1 \cap X_2));$$
$$(X_1, \widetilde{B}_1) \vee (X_2, \widetilde{B}_2) = (g \circ f(X_1 \cup X_2), \widetilde{B}_1 \cap \widetilde{B}_2)$$

Table 1
A formal fuzzy context.

$\widetilde{I}$	a	b	c	d	e
x_1	0.5	0.7	0.7	0.5	0.7
x_2	0.6	0.7	1.0	0.5	1.0
x_3	1.0	0.9	1.0	0.1	1.0
x_4	1.0	0.9	0.9	0.1	0.9
x_5	0.6	0.7	1.0	0.1	1.0

Table 2
All crisp-fuzzy concepts derived from Table 1.

	(Objects, attributes)
FC_1	$(\{x_1, x_2, x_3, x_4, x_5\}, \{a^{0.5}, b^{0.7}, c^{0.7}, d^{0.1}, e^{0.7}\})$
FC_2	$(\{x_2, x_3, x_4, x_5\}, \{a^{0.6}, b^{0.7}, c^{0.9}, d^{0.1}, e^{0.9}\})$
FC_3	$(\{x_1, x_2\}, \{a^{0.5}, b^{0,7}, c^{0.7}, d^{0.5}, e^{0.7}\})$
FC_4	$(\{x_2, x_3, x_5\}, \{a^{0.6}, b^{0,7}, c^{1.0}, d^{0.1}, e^{1.0}\})$
FC_5	$(\{x_3, x_4\}, \{a^{1.0}, b^{0,9}, c^{0.9}, d^{0.1}, e^{0.9}\})$
FC_6	$(\{x_2\}, \{a^{0.6}, b^{0,7}, c^{1.0}, d^{0.5}, e^{1.0}\})$
FC_7	$(\{x_3\}, \{a^{1.0}, b^{0,9}, c^{1.0}, d^{0.1}, e^{1.0}\})$
FC_8	$(\emptyset, \{a^{1.0}, b^{1.0}, c^{1.0}, d^{1.0}, e^{1.0}\})$

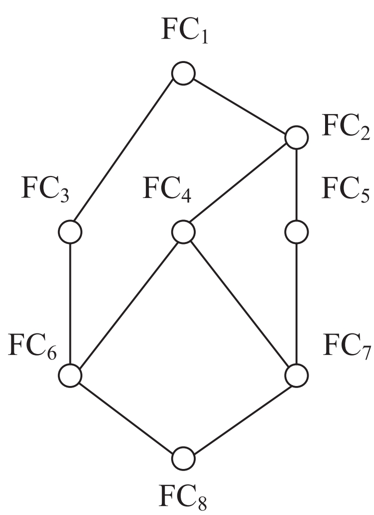

Fig. 1. The Hasse diagram of the concept lattice $L(U, \widetilde{A}, \widetilde{I})$.

$$= (g(\widetilde{B}_1 \cap \widetilde{B}_2), \widetilde{B}_1 \cap \widetilde{B}_2). \tag{2}$$

Example 1. Table 1 describes a formal fuzzy context $(U, A, \widetilde{I})$, where $U = \{x_1, x_2, x_3, x_4\}$, $A = \{a, b, c, d, e\}$, and the fuzzy relation $\widetilde{I}$ is defined in Table 1. All the crisp-fuzzy concepts derived from Table 1 are listed in Table 2, and the Hasse diagram of the concept lattice $L(U, \widetilde{A}, \widetilde{I})$ is depicted in Fig. 1.

2.2. Fuzzy sub-contexts and corresponding concept lattices

In this subsection, we discuss the relationship between the derivation operators on a formal fuzzy context and those of its sub-contexts.

Definition 2. Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context. For any $C \subseteq A$, we can obtain a formal fuzzy context $\mathbb{K}_C = (U, C, \widetilde{I}_C)$, which is called a sub-context of $\mathbb{K}$, where $\widetilde{I}_C = \widetilde{I} \cap (U \times C)$. For $X \in \mathcal{P}(U)$ and $\widetilde{B} \in \mathcal{F}(C)$, the operators $f_C : \mathcal{P}(U) \longrightarrow \mathcal{F}(C)$ and $g_C : \mathcal{F}(C) \longrightarrow \mathcal{P}(U)$ are defined respectively as follows:

$$f_C(X)(a) = \bigwedge_{x \in X} \widetilde{I}(x, a), \quad a \in C,$$
$$g_C(\widetilde{B}) = \{x \in U | \forall\ a \in C, \widetilde{B}(a) \leq \widetilde{I}(x, a)\}. \tag{3}$$

For simplicity, we use $f(X)$ and $g(\widetilde{B})$ instead of $f_A(X)$ and $g_A(\widetilde{B})$, respectively.

Table 3
A sub-context $(U, C, \widetilde{I}_C)$.

$\widetilde{I}$	a	c	d
x_1	0.5	0.7	0.5
x_2	0.6	1.0	0.5
x_3	1.0	1.0	0.1
x_4	1.0	0.9	0.1
x_5	0.6	1.0	0.1

Let A be a nonempty finite universe of discourse. For any $C \subseteq A$, its characteristic function $\mathcal{X}_C$ is defined by

$$\mathcal{X}_C(a) = \begin{cases} 1, & a \in C, \\ 0, & a \notin C. \end{cases}$$

Property 2. *Let $(U, A, \widetilde{I})$ be a formal fuzzy context and $C \subseteq A$. Then, $\forall X \subseteq U$,*

(1) $f_C(X) = f(X) \cap \mathcal{X}_C$,
(2) $g \circ f(X) \subseteq g_C \circ f_C(X)$.

Proof. Item (1) follows immediately from Definitions 1 and 2.
(2) For any $x \in g \circ f(X)$, by item (1) we have

$$\begin{aligned} x \in g \circ f(X) &\Leftrightarrow \forall\ a \in A, \quad f(X)(a) \leq \widetilde{I}(x, a) \\ &\Rightarrow \forall\ a \in C, \quad f_C(X)(a) \leq \widetilde{I}(x, a) \\ &\Leftrightarrow x \in g_C \circ f_C(X). \end{aligned}$$

Therefore, $g \circ f(X) \subseteq g_C \circ f_C(X)$. □

Example 2. In Example 1, let $C = \{a, c, d\}$. Then $(U, C, \widetilde{I}_C)$ is a sub-context of $(U, A, \widetilde{I})$, and is represented by Table 3. For any $X \subseteq U$, it can be easily checked that $f_C(X) = f(X) \cap \mathcal{X}_C$.

Lemma 1. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context, $C \subset A$, $X \subseteq U$, and $X_i \subseteq U, i = 1, 2, \ldots, k$. If*

$$f(X) = \bigcap_{i=1}^{k} f(X_i),$$

then

$$f_C(X) = \bigcap_{i=1}^{k} f_C(X_i).$$

Proof.

$$\begin{aligned} f_C(X) &= f(X) \cap C \\ &= \left(\bigcap_{i=1}^{k} f(X_i) \right) \cap C \\ &= \bigcap_{i=1}^{k} (f(X_i) \cap C) \\ &= \bigcap_{i=1}^{k} f_C(X_i). \end{aligned}$$

□

Lemma 1 says that if the intension derived from an object set can be represented as the intersection of intensions derived from some object sets in a formal fuzzy context, then in any sub-context, the equality still holds.

Corollary 1. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context, $C \subset A$, $x \in U$, and $x_i \in U, i = 1, 2, \ldots, k$. If*

$$f(x) = \bigcap_{i=1}^{k} f(x_i),$$

then

$$f_C(X) = \bigcap_{i=1}^{k} f_C(X_i).$$

Lemma 2. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context, $C \subset A$, $X \subseteq U$, and $X_i \subseteq U, i = 1, 2, \ldots, k$. If*

$$(g(f(X)), f(X)) = \bigvee_{i=1}^{k} (g(f(X_i)), f(X_i)), \tag{4}$$

then

$$(g_C(f_C(X)), f_C(X)) = \bigvee_{i=1}^{k} (g_C(f_C(X_i)), f_C(X_i)). \tag{5}$$

Proof. From Eq. (4), we have

$$f(X) = \bigcap_{i=1}^{k} f(X_i).$$

Then by Lemma 1, we have

$$f_C(X) = \bigcap_{i=1}^{k} f_C(X_i). \tag{6}$$

Hence,

$$\begin{aligned} g_C \circ f_C\left(\bigcup_{i=1}^{k} (g_C \circ f_C(X_i))\right) &= g_C\left(\bigcap_{i=1}^{k} (f_C \circ g_C \circ f_C(X_i))\right) \\ &= g_C\left(\bigcap_{i=1}^{k} f_C(X_i)\right) \\ &= g_C \circ f_C(X). \end{aligned} \tag{7}$$

By Eqs. (6) and (7), we conclude Eq. (5). □

Lemma 2 says that if a crisp-fuzzy concept derived from an object set is a join of some crisp-fuzzy concepts derived from a finite object sets, then in any sub-context, the corresponding crisp-fuzzy concept derived from the same object set can also be represented as the join of crisp-fuzzy concepts derived from these object sets.

Corollary 2. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context, $C \subset A$, $x \in U$, and $x_i \in U, i = 1, 2, \ldots, k$. If*

$$(g(f(x)), f(x)) = \bigvee_{i=1}^{k} (g(f(x_i)), f(x_i)), \tag{8}$$

then

$$(g_C(f_C(x)), f_C(x)) = \bigvee_{i=1}^{k} (g_C(f_C(x_i)), f_C(x_i)). \tag{9}$$

3. Join-irreducible elements

It is known that irreducible element plays an important role in computing the attribute reduction in a formal context. In this section, we study the properties of join-irreducible elements derived from a formal fuzzy context and its sub-contexts.

Definition 3 [11]**.** Let L be a finite lattice and $v \in L$. We denote

$$v_* = \bigvee \{x \in L |\ x < v\}.$$

v_* is said to be join-irreducible if $v \neq v_*$.

Theorem 1 [11]**.** *Let L be a finite lattice. Every element in L is a join of some join-irreducible elements.*

It should be noted that every crisp-fuzzy concept $(X, \widetilde{B})$ in the concept lattice $L(U, \widetilde{A}, \widetilde{I})$ can be represented as a join of object concepts of its extension, that is,

$$(X, \widetilde{B}) = \bigvee_{x \in X} (g \circ f(x), f(x)).$$

Theorem 2. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context, $C \subseteq A$ and $x \in U$. If $(g_C(f_C(x)), f_C(x))$ is a join-irreducible element in $L(U, \widetilde{C}, \widetilde{I}_C)$, then $(g(f(x)), f(x))$ is also a join-irreducible element in $L(U, \widetilde{A}, \widetilde{I})$.*

Proof. If $(g(f(x)), f(x))$ is not a join-irreducible element in $L(U, \widetilde{A}, \widetilde{I})$, then $(g(f(x)), f(x))$ is a join of some join-irreducible elements of $L(U, \widetilde{A}, \widetilde{I})$, i.e., there exists $x_i \in U, i = 1, 2, \ldots, k$ $(k \geq 2)$ such that

$$(g(f(x)), f(x)) = \bigvee_{i=1}^{k} (g(f(x_i)), f(x_i)),$$

where $(g(f(x_i)), f(x_i))$ $(i = 1, 2, \ldots, k)$ is join-irreducible element.

By Corollary 2, we have

$$(g_C(f_C(x)), f_C(x)) = \bigvee_{i=1}^{k} (g_C(f_C(x_i)), f_C(x_i)).$$

We conclude that $(g_C(f_C(x)), f_C(x))$ can be represented as a join of some join-irreducible elements of $L(U, \widetilde{C}, \widetilde{I}_C)$, which contradicts the assumption that $(g_C(f_C(x)), f_C(x))$ is a join-irreducible element in $L(U, \widetilde{C}, \widetilde{I}_C)$. Thus, we have proved that $(g(f(x)), f(x))$ is a join-irreducible element in $L(U, \widetilde{A}, \widetilde{I})$. □

Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context and $C \subseteq A$. A binary relation R_C is defined by

$$R_C = \{(x, y) \in U \times U | \widetilde{I}(x, a) \leq \widetilde{I}(y, a), \forall a \in C\}, \tag{10}$$

R_C is called an ordered relation on the object set, where $(x, y) \in R_C$ means that y is not less than x with respect to all attributes in C. It is evident that

$$R_C = \bigcap_{a \in C} R_{\{a\}}.$$

For any $x \in U$, its granule of knowledge induced by the ordered relation R_C is

$$\begin{aligned} [x]_{R_C} &= \{y \in U |\ (x, y) \in R_C\} \\ &= \{y \in U |\ \widetilde{I}(x, a) \leq \widetilde{I}(y, a), \forall\ a \in C\}, \end{aligned} \tag{11}$$

where $[x]_{R_C}$ is the set of objects whose attribute value is not less than x with respect to all attributes in C.

Lemma 3. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context, $C \subseteq A$ and $x \in U$. Then,*

$$f_C([x]_{R_C})(a) = f_C(x)(a), \ \ \forall a \in C.$$

Proof. By Eq. (3), we obtain

$$\begin{aligned} f_C([x]_{R_C})(a) &= \bigwedge_{x \in [x]_{R_C}} \widetilde{I}(x, a) \\ &= \widetilde{I}(x, a) \\ &= f_C(x)(a). \end{aligned}$$

□

Theorem 3. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context, $C \subseteq A$ and $x \in U$. Then, $([x]_{R_C}, f_C([x]_{R_C}))$ is a crisp-fuzzy formal concept of $\mathbb{K}_C$ and*

$$[x]_{R_C} = g_C \circ f_C(x).$$

Proof. From Lemma 3, we obtain

$$g_C \circ f_C([x]_{R_C}) = g_C \circ f_C(x).$$

By Property 1 (2), we have

$$[x]_{R_C} \subseteq g_C \circ f_C([x]_{R_C}) = g_C \circ f_C(x). \tag{12}$$

On the other hand, for any $y \in g_C \circ f_C([x]_{R_C})$, we have $\{y\} \subseteq g_C \circ f_C(x)$. By Property 1 (1) and (3), we conclude that

$$f_C(x) = f_C \circ g_C \circ f_C(x) \subseteq f_C(\{y\}) = f_C(y),$$

that is $f_C(x) \subseteq f_C(y)$, this means

$$f_C(x) = \widetilde{I}(x, a) \le \widetilde{I}(y, a) = f_C(y), \forall\ a \in C.$$

Hence, $y \in [x]_{R_C}$, and from which we obtain

$$g_C \circ f_C([x]_{R_C}) \subseteq [x]_{R_C}. \tag{13}$$

It follows from Eqs. (12) and (13) that

$$g_C \circ f_C([x]_{R_A}) = g_C \circ f_C(x) = [x]_{R_C},$$

and $([x]_{R_C}, f_C([x]_{R_C}))$ is a crisp-fuzzy formal concept. $\square$

Let $(U, A, \widetilde{I})$ be a formal fuzzy context and $C \subseteq A$. We denote

$$\beta(U) = \{([x]_{R_C}, f([x]_{R_C}))|\ x \in U\},$$
$$\gamma(U) = \{(g_C \circ f_C(x), f_C(x))|\ x \in U\}.$$

Corollary 3. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context and $C \subseteq A$. Then $\beta(U) = \gamma(U)$.*

Proof. It can be straightforwardly derived from Theorem 3. $\square$

Definition 4. Let $(U, \le)$ be a partial order set, $x, y \in U$ and $x < y$. x is called the lower close neighbor of y, if there does not exist $z \in U$ such that $x < z < y$. Here, y is also called the upper close neighbor of x and is denoted by $x \prec y$.

For any $y \in U$, we denote $\alpha(y) = \{x \in U | x \prec y\}$. Note that $\alpha(y)$ is the set of lower close neighbors of y. In the discussion to follow, for simplicity, we use the symbol $|\cdot|$ to denote the cardinality of a set.

Theorem 4. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context and $x \in U$. Then object concept $([x]_{R_A}, f([x]_{R_A}))$ is a join-irreducible element in $L(U, \widetilde{A}, \widetilde{I})$ iff $|\alpha(([x]_{R_A}, f([x]_{R_A})))| \le 1$.*

Proof. $(\Rightarrow)$ Suppose that $|\alpha(([x]_{R_A}, f([x]_{R_A})))| \ge 2$. Let

$$([z]_{R_A}, f([z]_{R_A})), ([u]_{R_A}, f([u]_{R_A})) \in \alpha(([x]_{R_A}, f([x]_{R_A}))).$$

We are going to prove that

$$([z]_{R_A}, f([z]_{R_A})) \vee ([u]_{R_A}, f([u]_{R_A})) = ([x]_{R_A}, f([x]_{R_A})).$$

Notice that

$$\begin{aligned} ([x]_{R_A}, f([x]_{R_A})) &\ge ([z]_{R_A}, f([z]_{R_A})) \vee ([u]_{R_A}, f([u]_{R_A})) \\ &= (g \circ f([z]_{R_A} \cup [u]_{R_A}), f([z]_{R_A}) \cap f([u]_{R_A})) \\ &> ([z]_{R_A}, f([z]_{R_A})) \text{ and } ([u]_{R_A}, f([u]_{R_A})). \end{aligned}$$

If

$$(g \circ f([z]_{R_A} \cup [u]_{R_A}), f([z]_{R_A}) \cap f([u]_{R_A})) < ([x]_{R_A}, f([x]_{R_A})),$$

then it is in conflict with Definition 4. Thus,

$$(g \circ f([z]_{R_A} \cup [u]_{R_A}), f([z]_{R_A}) \cap f([u]_{R_A})) = ([x]_{R_A}, f([x]_{R_A})).$$

It also conflicts with the assumption that $([x]_{R_A}, f([x]_{R_A}))$ is a join-irreducible element. Consequently, we conclude that $|\alpha(([x]_{R_A}, f([x]_{R_A})))| \le 1$, i.e.,

$$\begin{aligned} &|\{([y]_{R_A}, f([y]_{R_A})) \in L(U, \widetilde{A}, \widetilde{I})|\ ([y]_{R_A}, f([y]_{R_A})) \\ &\quad \prec ([x]_{R_A}, f([x]_{R_A}))\}| \le 1. \end{aligned}$$

$(\Leftarrow)$ Assume that $|\alpha(([x]_{R_A}, f([x]_{R_A})))| \le 1$.

If $|\alpha(([x]_{R_A}, f([x]_{R_A})))| = 0$, then $([x]_{R_A}, f([x]_{R_A}))$ does not contain any sub-concept. Hence, by Definition 3, $([x]_{R_A}, f([x]_{R_A}))$ itself is a join-irreducible element.

If $|\alpha(([x]_{R_A}, f([x]_{R_A})))| = 1$, then $([x]_{R_A}, f([x]_{R_A}))$ has only one lower close neighbor, and we denote it as $([u]_{R_A}, f([u]_{R_A}))$. It is evident that

$$([u]_{R_A}, f([u]_{R_A})) < ([x]_{R_A}, f([x]_{R_A})).$$

By Definition 3, we conclude that $([x]_{R_A}, f([x]_{R_A}))$ is a join-irreducible element. $\square$

By Theorem 4 we can easily determine whether or not an object concept is a join-irreducible element.

Corollary 4. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context and $x \in U$. Then object concept $([x]_{R_A}, f([x]_{R_A}))$ is a join-irreducible element in $L(U, \widetilde{A}, \widetilde{I})$ iff $|\{[y]_{R_A} \in U/R_A|\ [y]_{R_A} \prec [x]_{R_A}\}| \le 1$.*

Proof. It can simply be proved from Theorems 3 and 4. $\square$

Example 3. In Example 1, by computing we have $[x_1]_{R_A} = \{x_1, x_2\}$, $[x_2]_{R_A} = \{x_2\}$, $[x_3]_{R_A} = \{x_3\}$, $[x_4]_{R_A} = \{x_3, x_4\}$, $[x_5]_{R_A} = \{x_2, x_3, x_5\}$. Thus,

$$|\{[y]_{R_A} \in U/R_A|\ [y]_{R_A} \prec [x_1]_{R_A}\}| = 1,$$
$$|\{[y]_{R_A} \in U/R_A|\ [y]_{R_A} \prec [x_2]_{R_A}\}| = 0,$$
$$|\{[y]_{R_A} \in U/R_A|\ [y]_{R_A} \prec [x_3]_{R_A}\}| = 0,$$
$$|\{[y]_{R_A} \in U/R_A|\ [y]_{R_A} \prec [x_4]_{R_A}\}| = 1,$$

and $|\{[y]_{R_A} \in U/R_A|\ [y]_{R_A} \prec [x_5]_{R_A}\}| = 2$. Using Corollary 4 we conclude that $([x_1]_{R_A}, f([x_1]_{R_A}))$, $([x_2]_{R_A}, f([x_2]_{R_A}))$, $([x_3]_{R_A}, f([x_3]_{R_A}))$ and $([x_4]_{R_A}, f([x_4]_{R_A}))$ are join-irreducible elements in $L(U, \widetilde{A}, \widetilde{I})$.

4. Granular reducts of formal fuzzy contexts

Based on the notion of crisp-fuzzy formal concept, we first present some deterministic approaches to granular reducts of formal fuzzy contexts, and then discuss the reduction method and the corresponding algorithm.

Definition 5. Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context. An attribute subset $C \subseteq A$ is referred to as a granular consistent set of $\mathbb{K}$ if $g_C \circ f_C(x) = g \circ f(x)$ for all $x \in U$. If $C \subseteq A$ is a granular consistent set of $\mathbb{K}$ and there is no proper subset $D \subset C$ such that D is a granular consistent set, then C is referred to as a granular reduct of $\mathbb{K}$.

From Definition 5, we can see that a granular reduct is a minimal attribute set preserving all the object granules of a concept lattice.

Theorem 5. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context and $C \subseteq A$. Then C is a granular consistent set of $\mathbb{K}$ iff*

$$g_C \circ f_C(x) \subseteq g \circ f(x),\ \ \forall\ x \in U. \tag{14}$$

Proof. By Property 2 (4), we know that

$$g \circ f(x) \subseteq g_C \circ f_C(x), \forall\ x \in U.$$

Hence, we conclude that C is a granular consistent set if and only if Eq. (14) holds. $\square$

We denote the set of all granular reducts of $\mathbb{K}$ as $Red(\mathbb{K})$. According to the significance of the attributes, based on granular reducts, the attribute set A is divided into three parts:

- Indispensable attribute (core attribute) set $C_k : C_k = \bigcap Red(\mathbb{K})$;
- Relatively necessary attribute set $K_k : K_k = \bigcup Red(\mathbb{K}) - \bigcap Red(\mathbb{K})$;
- Unnecessary attribute set $I_k : I_k = A - \bigcup Red(\mathbb{K})$.

Corollary 5. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context and $a \in A$. Then a is an indispensable attribute iff there exists $x \in U$ such that*

$$g_{A-\{a\}} \circ f_{A-\{a\}}(x) \not\subseteq g \circ f(x). \tag{15}$$

Proof. It can easily be proved from Theorem 5 and the definition of indispensable attribute. □

Corollary 5 can help us determine whether or not an attribute is indispensable.

Definition 6. Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context and $(x, y) \in U \times U$. We define

$$\mathcal{D}(x, y) = \{a \in A \mid \widetilde{I}(x, a) > \widetilde{I}(y, a)\},$$

where $\mathcal{D}(x, y)$ is referred to as the granular discernibility attribute set of x and y, and $M = (\mathcal{D}(x, y) \mid (x, y) \in U \times U)$ is called the granular discernibility matrix of $\mathbb{K}$.

We denote

$$M_0 = \{\mathcal{D}(x, y) \mid \mathcal{D}(x, y) \neq \emptyset, (x, y) \in U \times U\}.$$

Theorem 6. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context and $C \subseteq A$. Then, C is a granular consistent set iff $C \cap \mathcal{D}(x, y) \neq \emptyset$ for all $\mathcal{D}(x, y) \in M_0$.*

Proof. ($\Rightarrow$) Let C be a granular consistent set. By Theorem 5, we have $g_C \circ f_C(x) \subseteq g \circ f(x)$ for all $x \in U$. Using Theorem 3 we obtain

$$[x]_{R_C} \subseteq [x]_{R_A}, \ \forall\ x \in U. \tag{16}$$

For any $\mathcal{D}(x, y) \in M_0$, there exists $a \in A$ such that $\widetilde{I}(x, a) > \widetilde{I}(y, a)$. Hence, $y \notin [x]_{R_A}$. By Eq. (16), we have $y \notin [x]_{R_C}$. Thus, there exists $c \in C$ such that $\widetilde{I}(x, c) > \widetilde{I}(y, c)$. By Definition 6, we conclude that $c \in \mathcal{D}(x, y)$. Hence, $C \cap \mathcal{D}(x, y) \neq \emptyset$.

($\Leftarrow$) Suppose that $C \cap \mathcal{D}(x, y) \neq \emptyset$ for all $D(x, y) \in M_0$. For any $x, y \in U$, if $y \notin [x]_{R_A}$, i.e. there exists $a \in A$ such that $\widetilde{I}(x, a) > \widetilde{I}(y, a)$, we have $\mathcal{D}(x, y) \neq 0$, hence $C \cap \mathcal{D}(x, y) \neq \emptyset$. Thus, there exists $c \in C$ such that $\widetilde{I}(x, c) > \widetilde{I}(y, c)$, which means $y \notin [x]_{R_C}$, and we conclude that $[x]_{R_C} \subseteq [x]_{R_A}$. Since $[x]_{R_C} = g_C \circ f_C(x)$ and $[x]_{R_A} = g \circ f(x)$, it follows that

$$g_C \circ f_C(x) \subseteq g \circ f(x).$$

By Theorem 5, we obtain that C is a granular consistent set of $\mathbb{K}$. □

Theorem 6 provides a method to determine whether or not an attribute set is consistent. By employing the granular discernibility matrix, we obtain the following judgment theorem of core attribute.

Theorem 7. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context and $a \in A$. Then, a is an indispensable (core) attribute in $\mathbb{K}$ iff there exists $(x, y) \in U \times U$ such that $\mathcal{D}(x, y) = \{a\}$.*

Proof. ($\Rightarrow$) Assume that a is an indispensable attribute in $\mathbb{K}$, then $A - \{a\}$ is not a granular consistent set of $\mathbb{K}$. By Theorem 6 we have $g_{A-\{a\}} \circ f_{A-\{a\}}(x) \nsubseteq g \circ f(x)$, namely, $[x]_{R_{A-\{a\}}} \nsubseteq [x]_{R_A}$. Thus, $[x]_{R_A} \subset [x]_{R_{A-\{a\}}}$. Hence, there exists $y \in U$ such that $y \in [x]_{R_{A-\{a\}}}$ and $y \notin [x]_{R_A}$, which means $\widetilde{I}(x, b) \leq \widetilde{I}(y, b)$ $(\forall\ b \in A - \{a\})$ and $\widetilde{I}(x, a) > \widetilde{I}(y, a)$. Therefore, by Definition 6 we conclude that $\mathcal{D}(x, y) = \{a\}$.

($\Leftarrow$) If there exists $(x, y) \in U \times U$ such that $\mathcal{D}(x, y) = \{a\}$, then, by Definition 6, we obtain $\widetilde{I}(x, b) \leq \widetilde{I}(y, b)$ $(\forall\ b \in A - \{a\})$ and $\widetilde{I}(x, a) > \widetilde{I}(y, a)$. Thus, $[x]_{R_{A-\{a\}}} \nsubseteq [x]_{R_A}$, that is, $g_{A-\{a\}} \circ f_{A-\{a\}}(x) \nsubseteq g \circ f(x)$. Hence, $a \in \bigcap Red(\mathbb{K})$. Therefore, a is an indispensable attribute in $\mathbb{K}$. □

Let $\bigvee \mathcal{D}(x, y)$ be a Boolean expression which is equal to 1, if $\mathcal{D}(x, y) = \emptyset$. Otherwise, $\bigvee \mathcal{D}(x, y)$ is a disjunction of variables corresponding to the attributes contained in $\mathcal{D}(x, y)$.

Let $\triangle = \bigwedge_{(x,\ y) \in\ U \times U} \bigvee \mathcal{D}(x, y)$, where $\bigwedge$ is conjunction of literals. $\triangle$ is called the discernibility function of $L(U, \widetilde{A}, \widetilde{I})$.

Table 4
The discernibility matrix $\triangle$.

$\mathcal{D}(x_i, y_j)$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	$\emptyset$	d	d	d
x_2	ace	$\emptyset$	d	cde	d
x_3	$abce$	ab	$\emptyset$	ce	ab
x_4	$abce$	ab	$\emptyset$	$\emptyset$	ab
x_5	ace	$\emptyset$	$\emptyset$	ce	$\emptyset$

Theorem 8. *Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context and $C \subseteq A$. Then, C is a granular reduct of $\mathbb{K}$ iff $\bigwedge_{c \in C} c$ is a prime implicant of the discernibility function $\triangle$.*

Proof. ($\Rightarrow$) Let $C \subseteq A$ be a granular reduct of $\mathbb{K}$. By Theorem 6, we have

$$C \cap \mathcal{D}(x, y) \neq \emptyset, \ \ \forall\ \mathcal{D}(x, y) \in M_0.$$

Then, there exists $\mathcal{D}(x, y) \in M_0$ such that $C \cap \mathcal{D}(x, y) = \{c\}$ for any $c \in C$. It follows that $\bigwedge_{c \in C} c$ is a prime implicant of the discernibility function.

($\Leftarrow$) Suppose $\bigwedge_{c \in C} c$ is a prime implicant of the discernibility function $\triangle$. Then

$$C \cap \mathcal{D}(x, y) \neq \emptyset, \ \ \forall\ \mathcal{D}(x, y) \in M_0,$$

and, there must exist $\mathcal{D}(x, y) \in M_0$ such that $(C - \{c\}) \cap \mathcal{D}(x, y) = \emptyset$. Therefore, we conclude that C is a granular reduct of $\mathbb{K}$. □

Let

$$\triangle = \bigwedge_{(x,\ y) \in\ U \times U} \bigvee \mathcal{D}(x, y) = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_s \right),$$

where $\bigwedge_{s=1}^{q_k} a_s$, $k \leq t$, are all the prime implicants of the discernibility function $\triangle$. We denote $N_k = \{a_s | s = 1, 2, \ldots, q_k\}$. Then $\{N_k | k = 1, 2, \ldots, t\}$ is the set of all granular reducts of $\mathbb{K}$.

Discernibility functions are monotonic Boolean functions and their prime implications uniquely determine all the granular reducts of formal fuzzy contexts.

By Theorem 8, the procedure for computing granular reducts is given in Algorithm 1. The complexity of line 1 is $|A||U|^2$ and the complexity from line 2 to line 3 is $2^{|A|}$. Hence, the maximum time complexity of Algorithm 1 is $t = 2^{|A|} + |A||U|^2$.

Algorithm 1 Computing all granular reducts of $(U, \widetilde{A}, \widetilde{I})$.

Input:
A formal fuzzy context $\mathbb{K} = (U, A, \widetilde{I})$.
Output:
$Red(\mathbb{K})$ // the set of granular reducts of $(U, \widetilde{A}, \widetilde{I})$.
1: Computing the granular discernibility matrix $M = (\mathcal{D}(x, y) \mid (x, y) \in U \times U)$;
2: Computing $\triangle = \bigwedge_{(x,y) \in U \times U} \bigvee \mathcal{D}(x, y)$;
3: Computing$\triangle = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_s \right)$;
4: Let $N_k = \{a_s | s \leq q_k\}$ and $Red(\mathbb{K}) = \{N_k | k \leq t\}$;
5: Return $Red(\mathbb{K})$.

Example 4. Continued from Example 1. The discernibility matrix of formal fuzzy context $\mathbb{K} = (U, A, \widetilde{I})$ is represented as Table 4. From Table 4, using discernibility function we have

$$\begin{aligned}
\triangle &= \bigwedge_{(x,\ y) \in\ U \times U} \bigvee \mathcal{D}(x, y) \\
&= d \wedge (a \vee b) \wedge (c \vee e) \wedge (a \vee c \vee e) \wedge (c \vee d \vee e) \wedge (a \vee b \vee c \vee e) \\
&= d \wedge (a \vee b) \wedge (c \vee e) \\
&= (a \wedge c \wedge d) \vee (b \wedge c \wedge d) \vee (a \wedge e \wedge d) \vee (b \wedge d \wedge e)
\end{aligned}$$

Table 5
A formal fuzzy decision context $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$.

	a	b	c	d	e	d_1	d_2	d_3
x_1	0.5	0.7	0.7	0.5	0.7	0.6	0.5	0.8
x_2	0.6	0.7	1.0	0.5	1.0	0.7	0.8	0.9
x_3	1.0	0.9	1.0	0.1	1.0	0.9	0.4	1.0
x_4	1.0	0.9	0.9	0.1	0.9	0.7	0.4	0.9
x_5	0.6	0.7	1.0	0.1	1.0	0.7	0.4	0.9

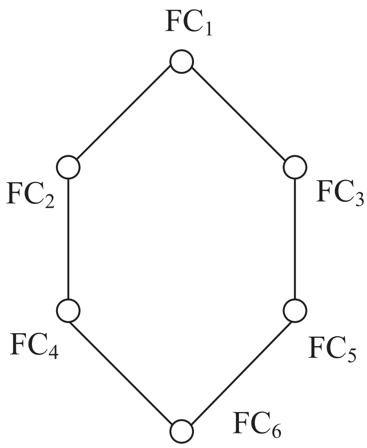

Fig. 2. The Hasse diagram of the concept lattice $L(U,\widetilde{D},\widetilde{J})$.

Hence, {a, c, d}, {b, c, d}, {a, e, d} and {b, d, e} are granular consistent sets of $\mathbb{K}$, and any proper subsets of {a, c, d}, {b, c, d}, {a, e, d} and {b, d, e} are not granular consistent sets. Therefore, {a, c, d}, {b, c, d}, {a, e, d} and {b, d, e} are granular reducts of $\mathbb{K}$, and d is an indispensable attribute. By the reduction, we have the reduced formal fuzzy contexts $(U,\{a,c,d\},\widetilde{I}_{\{a,c,d\}})$, $(U,\{b,c,d\},\widetilde{I}_{\{b,c,d\}})$, $(U,\{a,e,d\},\widetilde{I}_{\{a,e,d\}})$ and $(U,\{b,d,e\},\widetilde{I}_{\{b,d,e\}})$ respectively. We obtain the same number of object concepts from the reduced formal fuzzy contexts $(U,\{a,c,d\},\widetilde{I}_{\{a,c,d\}})$, $(U,\{b,c,d\},\widetilde{I}_{\{b,c,d\}})$, $(U,\{a,e,d\},\widetilde{I}_{\{a,e,d\}})$ and $(U,\{b,d,e\},\widetilde{I}_{\{b,d,e\}})$.

5. Granular reducts of consistent formal fuzzy decision contexts

In this section, we introduce the notion of a formal fuzzy decision context as an extension of formal fuzzy context by dividing the attributes into condition attributes and decision attributes. Similarly, we consider granular reducts and attribute characteristics of consistent formal fuzzy decision contexts.

Definition 7. A formal fuzzy decision context is a quintuple $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$, where $(U,A,\widetilde{I})$ and $(U,D,\widetilde{J})$ are formal fuzzy contexts, $A\cap D=\emptyset$, A and D are conditional attribute set and decision attribute set, respectively.

Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a formal fuzzy decision context and $C\subseteq A$. The operator f_C and g_C in fuzzy contexts $(U,A,\widetilde{I})$ are defined by Eq. (3). To avoid confusion, the corresponding derivation operators in context $(U,D,\widetilde{J})$ are denoted as f_D and g_D.

Example 5. Table 5 represents a formal fuzzy decision context $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$, where, $U=\{x_1,x_2,\ x_3,\ x_4,\ x_5\}$, $A=\{a,b,c,d,e\}$ and $D=\{d_1,d_2,d_3\}$. Fig. 2 is the Hasse diagram of concept lattice $L(U,\widetilde{D},\widetilde{J})$, and Table 6 lists all crisp-fuzzy concepts of $L(U,\widetilde{D},\widetilde{J})$.

Definition 8. Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a formal fuzzy decision context. $\mathbb{S}$ is said to be consistent if $f\circ g(x)\subseteq f_D\circ g_D(x)$ for all $x\in U$. Otherwise, it is said to be inconsistent.

Table 6
All crisp-fuzzy concepts of $L(U,\widetilde{D},\widetilde{J})$.

	(objects, attributes)
FC_1	$(\{x_1,x_2,x_3,x_4,x_5\},\ \{d_1^{0.6},d_2^{0.4},d_3^{0.8}\})$
FC_2	$(\{x_1,x_2\},\ \{d_1^{0.6},d_2^{0.5},d_3^{0.8}\})$
FC_3	$(\{x_2,x_3,x_4,x_5\},\ \{d_1^{0.7},d_2^{0.4},d_3^{0.9}\})$
FC_4	$(\{x_2\},\ \{d_1^{0.7},d_2^{0.8},d_3^{0.9}\})$
FC_5	$(\{x_3\},\ \{d_1^{0.9},d_2^{0.4},d_3^{1.0}\})$
FC_6	$(\emptyset,\ \{d_1^{1.0},d_2^{1.0},d_3^{1.0}\})$

Example 6. In Example 5, since

$$g\circ f(x_1)=\{x_1,x_2\}=g_D\circ f_D(x_1),$$
$$g\circ f(x_2)=\{x_2\}=g_D\circ f_D(x_2),$$
$$g\circ f(x_3)=\{x_3\}=g_D\circ f_D(x_3),$$
$$g\circ f(x_4)=\{x_3,x_4\}\subseteq\{x_2,x_3,x_4,x_5\}=g_D\circ f_D(x_4),$$
$$g\circ f(x_5)=\{x_2,x_3,x_5\}\subseteq\{x_2,x_3,x_4,x_5\}=g_D\circ f_D(x_5),$$

we can see that $\mathbb{S}$ is a consistent formal fuzzy decision context.

Definition 9. Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a consistent formal fuzzy decision context and $C\subseteq A$. If $f_C\circ g_C(x)\subseteq f_D\circ g_D(x)$ for all $x\in U$, then C is referred to as a granular consistent set of $\mathbb{S}$. If C is a granular consistent set of $\mathbb{S}$ and no proper subset of C is a granular consistent set, then C is referred to as a granular reduct of $\mathbb{S}$.

We denote the set of all granular reducts of $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ as $Red(\mathbb{S})$. Similarly, the attribute set A is divided into three parts according to the significance of the attributes:

- Indispensable attribute (core attribute) set C_s : $C_s=\bigcap Red(\mathbb{S})$;
- Relatively necessary attribute set K_s : $K_s=\bigcup Red(\mathbb{S})-\bigcap Red(\mathbb{S})$;
- Unnecessary attribute set I_s : $I_s=A-\bigcup Red(\mathbb{S})$.

Similar to the definition of ordered relation R_C defined by Eq. (8), the ordered relation R_D with respect to the decision attribute set D in $(U,A,\widetilde{I},D,\widetilde{J})$ is defined by

$$R_D=\{(x,y)\in U\times U|\widetilde{J}(x,d)\le\widetilde{J}(y,d),\forall d\in D\}.$$

Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a consistent formal fuzzy decision context and $(x,\ y)\in U\times U$. We denote

$$\mathcal{D}^S(x,y)=\begin{cases}\{a\in A|\widetilde{I}(x,a)>\widetilde{I}(y,a)\}, & \widetilde{J}(x,d)>\widetilde{J}(y,d)\ (\exists\ d\in D);\\ \emptyset, & \widetilde{J}(x,d)\le\widetilde{J}(y,d)\ (\forall\ d\in D).\end{cases}$$

$\mathcal{D}^S(x,y)$ is referred to as the discernibility attribute set of x and y, and $M^S=(\mathcal{D}^S(x,y)\ |\ x,y\in U)$ is called the discernibility matrix of $\mathbb{S}$.

We denote

$$M_0^S=\{D^S(x,y)|\ \mathcal{D}^S(x,y)\neq\emptyset\ (x,y\in U)\}.$$

Theorem 9. *Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a consistent formal fuzzy decision context and $C\subseteq A$. Then, C is a granular consistent set iff $C\cap\mathcal{D}^S(x,y)\neq\emptyset$ for all $\mathcal{D}^S(x,y)\in M_0^S$.*

Proof. ($\Rightarrow$) For any $\mathcal{D}^S(x,y)\in\mathcal{D}_0$, from the above definition we conclude that $y\notin[x]_{R_D}$. Since C is a granular consistent set, we have

$$[x]_{R_C}=g_C\circ f_C(x)\subseteq g_D\circ f_D(x)=[x]_{R_D}.$$

Hence, $y\notin[x]_{R_C}$, that is, there exists an attribute $c\in C$ such that $\widetilde{I}(x,c)>\widetilde{I}(y,c)$, which implies that $c\in\mathcal{D}^S(x,y)$. Therefore, $c\in C\cap\mathcal{D}^S(x,y)$. It is evident that $C\cap\mathcal{D}^S(x,y)\neq\emptyset$.

($\Leftarrow$) Suppose that $C\cap\mathcal{D}^S(x,y)\neq\emptyset$ for all $\mathcal{D}^S(x,y)\in M_0^S$. Since $\mathbb{S}$ is a consistent formal fuzzy decision context, for any $x,\ y\in U$, if $y\notin[x]_{R_D}$, we have $y\notin[x]_{R_A}$, i.e., there exists $a\in A$ such that

Table 7
The discernibility matrix $\triangle$.

$\mathcal{D}^S(x,y)$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	$\emptyset$	d	d	d
x_2	ace	$\emptyset$	d	cde	d
x_3	$abce$	ab	$\emptyset$	ce	ab
x_4	$abce$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_5	ace	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

$\widetilde{I}(x,a) > \widetilde{I}(y,a)$. Hence, we conclude that $\mathcal{D}^S(x,y) \neq 0$. By the assumption, we obtain $C \cap \mathcal{D}^S(x,y) \neq \emptyset$. Thus, there exists $c \in C$ such that $\widetilde{I}(x,c) > \widetilde{I}(y,c)$, which means $y \notin [x]_{R_C}$, and we conclude that $[x]_{R_C} \subseteq [x]_{R_D}$. It follows that

$$[x]_{R_C} = g_C \circ f_C(x) \subseteq g_D \circ f_D(x) = [x]_{R_D}.$$

Therefore, we have proved that C is a granular consistent set of $\mathbb{S}$. □

Theorem 9 provides a method to determine whether or not an attribute set is consistent in $\mathbb{S}$.

By employing the granular discernibility matrix, we obtain straightforwardly the following judgment theorem of granular core attribute in a consistent formal fuzzy decision context.

Theorem 10. *Let $\mathbb{S} = (U, A, \widetilde{I}, D, \widetilde{J})$ be a consistent formal fuzzy decision context and $a \in A$. Then, a is an indispensable (core) attribute in $\mathbb{S}$ iff there exists $(x, y) \in U \times U$ such that $\mathcal{D}^S(x,y) = \{a\}$.*

Proof. ($\Rightarrow$) Assume that a is an indispensable attribute in $\mathbb{S}$, then $A - \{a\}$ is not a granular consistent set of $\mathbb{S}$. By Theorem 5, we have $g_{A-\{a\}} \circ f_{A-\{a\}}(x) \nsubseteq g_D \circ f_D(x)$, i.e., $[x]_{R_{A-\{a\}}} \nsubseteq [x]_{R_D}$. Thus, there exists $y \in U$ such that $y \in [x]_{R_{A-\{a\}}}$ and $y \notin [x]_{R_D}$, which implies that

$$\widetilde{I}(x,a) > \widetilde{I}(y,a),\ \ \widetilde{I}(x,b) \leq \widetilde{I}(y,b),\ \ \forall\ b \in A - \{a\},$$

and

$$\widetilde{J}(x,d) > \widetilde{J}(y,d)\ \ (\exists\ d \in D).$$

By the definition of $\mathcal{D}^S(x,y)$, we conclude that $\mathcal{D}^S(x,y) = \{a\}$.

($\Leftarrow$) If there exists $(x, y) \in U \times U$ such that $\mathcal{D}^S(x,y) = \{a\}$, then by definition we obtain that

$$\widetilde{I}(x,b) \leq \widetilde{I}(y,b)\ \ (\forall\ b \in A - \{a\}), \widetilde{I}(x,a) > \widetilde{I}(y,a)$$

and

$$\widetilde{J}(x,d) > \widetilde{J}(y,d)\ \ (\exists\ d \in D).$$

Thus, $y \in [x]_{R_{A-\{a\}}}$ and $y \notin [x]_{R_D}$, and we conclude that $[x]_{R_{A-\{a\}}} \nsubseteq [x]_{R_D}$, that is, $g_{A-\{a\}} \circ f_{A-\{a\}}(x) \nsubseteq g_D \circ f_D(x)$. Hence, $a \in \bigcap Red(\mathbb{S})$. Consequently, a is an indispensable attribute in $\mathbb{S}$. □

Let

$$\triangle^S = \bigwedge_{(x,\ y) \in\ U \times U} \bigvee \mathcal{D}^S(x,y) = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_s \right),$$

where $\bigwedge_{s=1}^{q_k} a_s$, $k \leq t$, are all the prime implicants of the discernibility function $\triangle^S$. Then $N_k^S = \{a_s | s \leq q_k\}, k \leq t$, are all granular reducts of $\mathbb{S}$.

By Theorem 9, the procedure for computing granular reducts is given in Algorithm 2. The complexity of line 1 is $|A||U|^2$ and the complexity from line 2 to line 3 is $2^{|A|}$. Hence, the maximum time complexity of Algorithm 1 is $t = 2^{|A|} + |A||U|^2$.

Example 7. Continued from Example 5. The discernibility matrix of the formal fuzzy decision context $\mathbb{S}$ is represented by Table 7. From Table 7, using the discernibility function we have:

Algorithm 2 Computing all granular reducts of a consistent formal fuzzy decision context $(U, A, \widetilde{I}, D, \widetilde{J})$.

Input:
A consistent formal fuzzy decision context $\mathbb{S} = (U, A, \widetilde{I}, D, \widetilde{J})$.
Output:
$RED(\mathbb{S})$ // the set of attribute reducts of $\mathbb{S}$.
1: Computing the granular discernibility matrix $M^S = (\mathcal{D}^S(x,y) \mid (x,y) \in U \times U)$;
2: Computing $\triangle^S = \bigwedge_{(x,y) \in U \times U} \bigvee \mathcal{D}^S(x,y)$;
3: Computing$\triangle^S = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_s \right)$;
4: Let $N_k^S = \{a_s | s \leq q_k\}$ and $Red(\mathbb{S}) = \{N_k^S | k \leq t\}$;
5: Return $Red(\mathbb{S})$.

$$\begin{aligned}\triangle^S &= \bigwedge_{(x,\ y) \in\ U \times U} \bigvee \mathcal{D}^S(x,y) \\ &= d \wedge (a \vee b) \wedge (c \vee e) \wedge (a \vee c \vee e) \wedge \\ &\quad (c \vee d \vee e) \wedge (a \vee b \vee c \vee e) \\ &= d \wedge (a \vee b) \wedge (c \vee e) \\ &= (a \wedge c \wedge d) \vee (b \wedge c \wedge d) \vee (a \wedge e \wedge d) \vee (b \wedge d \wedge e).\end{aligned}$$

Therefore, $\{a, c, d\}$, $\{b, c, d\}$, $\{a, e, d\}$ and $\{b, d, e\}$ are granular reducts of $\mathbb{S}$, and d is an indispensable attribute.

6. Relation between granular reduct and classification reduct in a formal fuzzy context

A formal fuzzy context can also be regarded as an information table, it is natural to consider attribute reducts of a fuzzy information table. Recently, by using rough set (RS) [36] approach, many types of attribute reducts have been proposed in fuzzy information tables from perspective of classification of the universe or rule-preserving [5,14,58].

For a formal fuzzy context, one can consider two types of reducts: RS reducts (based on classification or rule-preserving) and FCA reducts (based on concepts and their hierarchies). Hence, what exactly is the relation between FCA reducts and RS reducts is an interesting question. In this section, we consider attribute reducts of formal fuzzy contexts from the perspective of ordered relation and then discuss the relationship between granular reduct and classification reduct.

Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context and $C \subseteq A$. The ordered relation R_C is defined by Eq. (8). Then $\mathbb{K} = (U, A, \widetilde{I})$ can be treated as an ordered fuzzy information system. By U/R_C we denote the family set $\{[x]_{R_C} \mid x \in U\}$. Any element in U/R_C is called an ordered class. One can easily obtain the following properties:

- R_C is reflexive, transitive, and asymmetric;
- if $C \subseteq B \subseteq A$, then $R_C \supseteq R_B \supseteq R_A$;
- if $C, B \subseteq A$, then $R_{C \cup B} = R_C \cap R_B$;
- if $C \subseteq B \subseteq A$, then $[x]_{R_C} \supseteq [x]_{R_B} \supseteq [x]_{R_A}$;
- if $y \in [x]_{R_C}$, then $[y]_{R_C} \subseteq [x]_{R_C}$ and $[x]_{R_C} = \bigcup\{[y]_{R_C} : y \in [x]_{R_C}\}$;
- $[x]_{R_C} = [y]_{R_C}$ iff $\widetilde{I}(x,a) = \widetilde{I}(y,a) (\forall\ a \in C)$;
- $\mathcal{J} = \{[x]_{R_C} \mid x \in U\}$ constitutes a covering of U.

Let $\mathbb{K} = (U, A, \widetilde{I})$ be a formal fuzzy context. An attribute subset $C \subseteq A$ is called a classification consistent set (based on ordered relation) of $\mathbb{K}$ if $R_C = R_A$. Furthermore, if $R_{C-\{c\}} \neq R_A$ for all $c \in C$, then C is called a classification reduct (based on ordered relation) of $\mathbb{K}$.

We denote the set of all classification reducts of $\mathbb{K}=(U,A,\widetilde{I})$ as $Red(\mathbb{C})$. Similarly, based on the classification reduct, the attribute set A is divided into three parts:

- Indispensable attribute (core attribute) set $C_i : C_i = \bigcap Red(\mathbb{C})$;
- Relatively necessary attribute set $K_i : K_i = \bigcup Red(\mathbb{C}) - \bigcap Red(\mathbb{C})$;
- Unnecessary attribute set $I_i : I_i = A - \bigcup Red(\mathbb{C})$.

Theorem 11. *Let $\mathbb{K}=(U,A,\widetilde{I})$ be a formal fuzzy context and $a \in A$. Then, $a \in C_i$ iff $R_{A-\{a\}} \neq R_A$.*

Proof. ($\Rightarrow$) Let $a \in C_i$. If $R_{A-\{a\}} = R_A$, then there exists $C \subseteq A-\{a\}$ such that C is a classification reduct. Hence, we conclude that $a \notin \bigcap Red(\mathbb{C})$, which contradicts the assumption. Consequently, we have $R_{A-\{a\}} \neq R_A$.

($\Leftarrow$) It follows immediately from the definition of indispensable attribute. □

Theorem 12. *Let $\mathbb{K}=(U,A,\widetilde{I})$ be a formal fuzzy context and $a \in A$. Then, $a \in I_i$ iff $R_A = R_{A-\{a\}}$ and $R_{C_i} \subseteq R_{\{a\}}$ (where C_i is the set of indispensable attributes).*

Proof. ($\Rightarrow$) Suppose $a \in I_i$, then $a \notin C_i$. By Theorem 11, we have $R_A = R_{A-\{a\}}$. If $R_{C_i} \nsubseteq R_{\{a\}}$, then there exists $(x, y) \in U \times U$ such that $(x,y) \in R_{C_i}$ and $(x,y) \notin R_{\{a\}}$. Thus, we have $(x,y) \notin R_{C_i} \cap R_{\{a\}}$, i.e., $(x,y) \notin R_{C_i \cup \{a\}}$. Hence, $R_A \subseteq R_{C_i \cup \{a\}} \subset R_{C_i}$. It means that there exists a set $B \subseteq A$ such that $C_i \cup \{a\} \subseteq B$ and B is a granular reduct, which contradicts $a \in I_i$. Consequently, we have $R_{C_i} \subseteq R_{\{a\}}$.

($\Leftarrow$) Since $R_A = R_{A-\{a\}}$, form Theorem 11 we conclude that a is not an indispensable attribute, i.e., $a \notin C_i$. Assume that there exists $B \in Red(\mathbb{C})$ such that $a \in B$. It is evident that $C_i \subset B$. We denote $D = B - C_i \cup \{a\}$. Hence,

$$R_B = R_{C_i \cup \{a\} \cup D} = R_{C_i} \cap R_{\{a\}} \cap R_D.$$

From $R_{C_i} \subseteq R_{\{a\}}$, we obtain

$$R_{C_i} \cap R_{\{a\}} \cap R_D = R_{C_i} \cap R_D = R_{C_i \cup D} = R_{B-\{a\}}.$$

Thus, we have $R_B = R_{B-\{a\}}$, which contradicts $B \in Red(\mathbb{C})$. Consequently, $a \notin \bigcup Red(\mathbb{C})$. Therefore, we conclude that $a \in A - \bigcup Red(\mathbb{C}) = I_i$. □

Theorem 13. *Let $\mathbb{K}=(U,A,\widetilde{I})$ be a formal fuzzy context and $a \in A$. Then, $a \in K_i$ iff $R_A = R_{A-\{a\}}$ and $R_{C_i} \nsubseteq R_{\{a\}}$.*

Proof. It can easily be proved from Theorems 11 and 12. □

Theorem 14. *Let $\mathbb{K}=(U,A,\widetilde{I})$ be a formal fuzzy context and $C \subseteq A$. Then, C is a granular consistent set of $(U,\widetilde{A},\widetilde{I})$ iff C is a classification consistent set of $\mathbb{K}$.*

Proof. ($\Rightarrow$) Let C be a granular consistent set of $(U,\widetilde{A},\widetilde{I})$. Then $g_C \circ f_C(x) = g \circ f(x)$ for all $x \in U$. By Theorem 3, we obtain $[x]_{R_C} = [x]_{R_A}$ for all $x \in U$, which means $R_C = R_A$. It follows that C is a classification consistent set of $\mathbb{K}$.

($\Leftarrow$) Suppose C is a classification consistent set of $\mathbb{K}$, i.e., $R_A = R_C$. Then we have $[x]_{R_C} = [x]_{R_A}$ for all $x \in U$. From Theorem 3 and Definition 5, one can easily conclude that C is a granular consistent set. □

Corollary 6. *Let $\mathbb{K}=(U,A,\widetilde{I})$ be a formal fuzzy context and $C \subseteq A$. Then, C is a granular reduct of $\mathbb{K}$ iff C is a classification reduct of $\mathbb{K}$.*

Proof. It follows immediately from Theorem 14. □

Theorem 14 and Corollary 6 say that a granular consistent set (reduct) is also a classification consistent set (reduct) in a formal fuzzy context, and vice versa.

Theorem 15. *Let $\mathbb{K}=(U,A,\widetilde{I})$ be a formal fuzzy context and $a \in A$. Then,*

(1) $a \in C_k \Leftrightarrow a \in C_i$;
(2) $a \in I_k \Leftrightarrow a \in I_i$;
(3) $a \in K_k \Leftrightarrow a \in K_i$.

Proof. It can be proved directly from Corollary 6. □

By Theorem 15, we obtain that granular reducts and classification reducts in a formal fuzzy context have the same attribute characteristics. Thus, one can obtain all granular attribute characteristics via the classification attribute characteristics in a formal fuzzy context.

Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a consistent formal fuzzy decision context and $C \subseteq A$. A is called a classification consistent set of $\mathbb{S}$ if $R_C \subseteq R_D$. Furthermore, if $R_{C-\{c\}} \neq R_D$ for all $c \in C$, then C is called a classification reduct of $\mathbb{S}$.

The set of all classification reducts of $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ is denoted by $Red(\mathbb{D})$. Similarly, based on the classification reduct, the attribute set A is divided into three parts:

- Indispensable attribute (core attribute) set $C_d : C_d = \bigcap Red(\mathbb{D})$;
- Relatively necessary attribute set $K_d : K_d = \bigcup Red(\mathbb{D}) - \bigcap Red(\mathbb{D})$;
- Unnecessary attribute set $I_d : I_d = A - \bigcup Red(\mathbb{D})$.

Theorem 16. *Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a consistent formal fuzzy decision context and $a \in A$. Then, a is an indispensable attribute in $\mathbb{S}$ iff $R_{A-\{a\}} \nsubseteq R_D$.*

Proof. It is similar to the proof of Theorem 11. □

Theorem 17. *Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a consistent formal fuzzy decision context and $a \in A$. Then, a is an unnecessary attribute in $\mathbb{S}$ iff $R_{A-\{a\}} \subseteq R_D$ and $R_{C_d} \subseteq R_D \cup R_{\{a\}}$ (where C_d is the set of indispensable attributes).*

Proof. It is similar to the proof of Theorem 12. □

Theorem 18. *Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a consistent formal fuzzy decision context and $a \in A$. Then, a is a relatively necessary attribute in $\mathbb{S}$ iff $R_{A-\{a\}} \subseteq R_D$ and $R_{C_d} \nsubseteq R_D \cup R_{\{a\}}$.*

Proof. It is similar to the proof of Theorem 13. □

Theorem 19. *Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a consistent formal fuzzy decision context and $C \subseteq A$. Then, C is a granular consistent set of $\mathbb{S}$ iff C is a classification consistent set of $\mathbb{S}$.*

Proof. It is similar to the proof of Theorem 14. □

Corollary 7. *Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a consistent formal fuzzy decision context and $C \subseteq A$. Then, C is a granular reduct of $\mathbb{S}$ iff C is a classification reduct of $\mathbb{S}$.*

Proof. It can easily be proved from Theorem 19. □

Theorem 19 and Corollary 7 say that a granular consistent set (reduct) is also a classification consistent set (reduct) in a formal fuzzy decision context, and vice versa.

Theorem 20. *Let $\mathbb{S}=(U,A,\widetilde{I},D,\widetilde{J})$ be a consistent formal fuzzy decision context and $a \in A$. Then,*

(1) $a \in C_s \Leftrightarrow a \in C_d$;
(2) $a \in I_s \Leftrightarrow a \in I_d$;
(3) $a \in K_s \Leftrightarrow a \in K_d$.

Proof. It can easily be proved from Corollary 7. □

From Theorem 20, we know that granular reducts and classification reducts in a consistent formal fuzzy decision context have the same attribute characteristics. Thus, one can obtain all granular attribute characteristics via the classification attribute characteristics in a consistent formal fuzzy decision context.

7. Conclusion

It is known that the cost of constructing a concept lattice is a super-linear function of the corresponding context size and the efficient computing of concept lattices is of great importance. Thus, knowledge reduction plays a very crucial role in reducing the dimensionality of a context, especially in large databases. In this paper, we have proposed granular reduct approach in formal fuzzy contexts which can guarantee that the object granules obtained from the reduced formal fuzzy context are identical to those obtained from the initial formal fuzzy context. Since a granular reduct is a minimal attribute set preserving all the object concepts, one can formulate all the object concepts of the initial formal fuzzy context from one of its granular reduct. As a result, not only can the computational complexity of constructing the concept lattices be reduced, but also more concise representation of the concepts can be obtained. Thus, the mining of fuzzy decision rules is made more convenient after reduction.

For a formal fuzzy context, it is natural to consider two kinds of reduction, namely, granular reducts and classification reducts. In this paper, the relationship between granular reduct and classification reduct in a formal fuzzy context has been established, from which the following meaningful conclusions can be obtained: (1) A granular consistent set (reduct) is also a classification consistent set (reduct), and vice versa; (2) attribute characteristics in the classification reducts are identical to those in the granular reducts. From the above assertions, one can obtain all the granular reducts and their attribute characteristics via the classification reducts and their attribute characteristics, and vice versa.

In view of the future research, the mining of fuzzy decision rules in reduced formal fuzzy contexts will be further investigated. The relationship between the granular reducts and the classification reducts for inconsistent formal fuzzy decision contexts or interval-valued formal contexts deserves to be investigated.

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